Sampling

Guillaume Frèche

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Sampling allows the transform of an analog signal into a digital one. We describe this transform from both time and frequency perspectives.

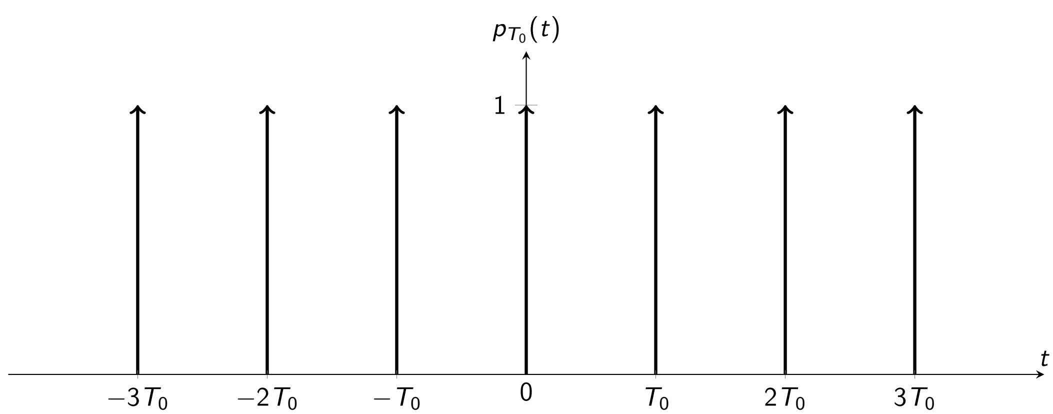
1 Time perspective

Definition 1.1 (Sampling)
Sampling is an application from $\mathcal{F}(\mathbb{R},\mathbb{K})$ to $\mathcal{F}(\mathbb{Z},\mathbb{K})$ which maps an analog signal x to the digital signal $(x(t_n))_{n\in\mathbb{Z}}$, where $(t_n)_{n\in\mathbb{Z}}$ is an increasing sequence of real numbers.
Periodic sampling with period $T_s > 0$ is the sampling corresponding to sequence $t_n = nT_s$, i.e. it maps an analog signal x to the digital signal $(x(nT_s))_{n\in\mathbb{Z}}$. In this case, the **sampling frequency** is the number $f_s = \dfrac{1}{T_s}$ and the **sampling impulse** is the number $\omega_s = 2\pi f_s = \dfrac{2\pi}{T_s}$.

In this lecture, we exclusively focus on periodic sampling.

Definition 1.2 (Dirac comb)
The **Dirac comb** with period $T_0 > 0$ is the distribution $p_{T_0} = \sum_{n\in\mathbb{Z}} \delta_{nT_0}$, i.e.

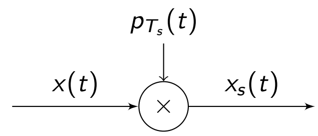
$$\forall t \in \mathbb{R} \qquad p_{T_0}(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT_0)$$

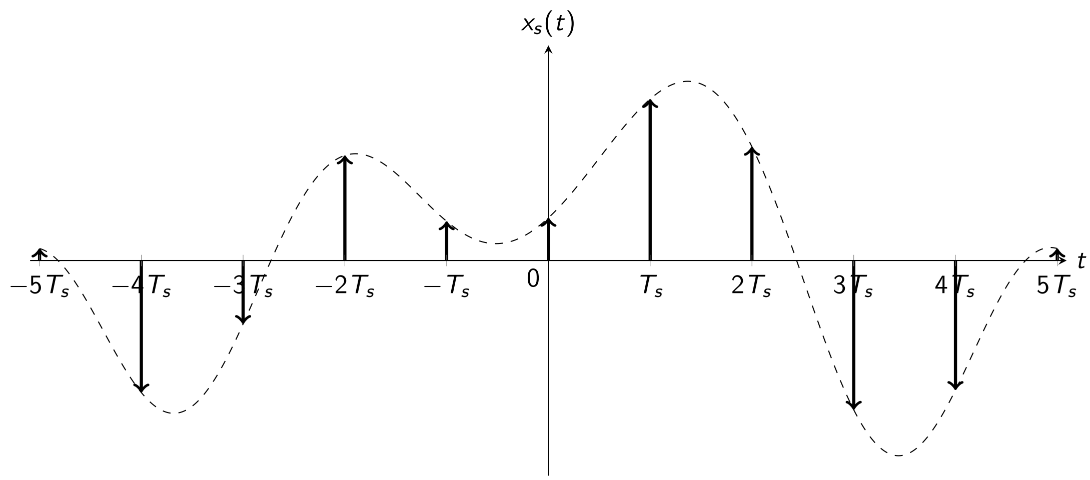


Definition 1.3 (Sampled signal)

The **sampled signal** with sampling period T_s of an analog signal x is the analog signal x_s obtained by multiplying x by the Dirac comb p_{T_s}:

$$\forall t \in \mathbb{R} \qquad x_s(t) = x(t)p_{T_s}(t) = \sum_{n=-\infty}^{+\infty} x(nT_s)\delta(t - nT_s)$$





Remarks:

- WARNING: As an infinite sum of shifted and delayed analog Dirac delta functions, the sampled signal is not a digital signal, but an analog one. We must apply on this signal the definitions and properties of analog signals.
- To obtain the digital signal x from the sampled signal x_s, we have to locally integrate the sampled signal, i.e.

$$\forall n \in \mathbb{Z} \qquad \forall \varepsilon \in]0, T_s[\qquad x[n] = \int_{n-\varepsilon}^{n+\varepsilon} x_s(t)dt$$

- With sampling, we can deduce the definition and properties of digital convolution from analog convolution of sampled signals. Indeed, let two sampled signals $x_s(t) = \sum_{n=-\infty}^{+\infty} x[n]\delta(t - nT_s)$ and $y_s(t) = \sum_{n=-\infty}^{+\infty} y[n]\delta(t - nT_s)$, and let

$$z_s(t) = (x_s * y_s)(t) = \sum_{n=-\infty}^{+\infty} z[n]\delta(t - nT_s)$$

Then by linearity of the analog convolution, for any $t \in \mathbb{R}$,

$$\begin{aligned} z_s(t) &= \left(\left(\sum_{n=-\infty}^{+\infty} x[n]\delta_{nT_s} \right) * \left(\sum_{m=-\infty}^{+\infty} y[m]\delta_{mT_s} \right) \right)(t) = \sum_{n=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} x[n]y[m] \left(\delta_{nT_s} * \delta_{mT_s} \right)(t) \\ &= \sum_{n=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} x[n]y[m]\delta\left(t - (n+m)T_s\right) = \sum_{n=-\infty}^{+\infty} \left(\sum_{k=-\infty}^{+\infty} x[k]y[n-k] \right) \delta(t - nT_s) \end{aligned}$$

which justifies the definition of digital convolution.

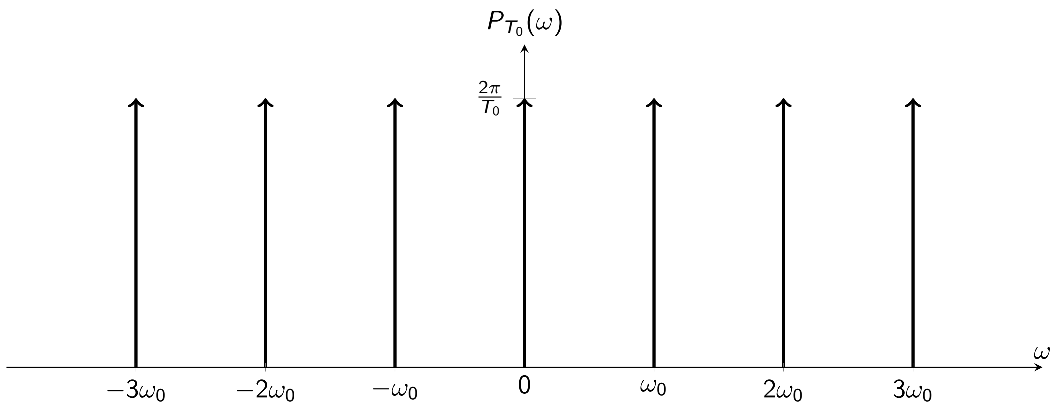
2 Frequency perspective

Now we study the effects of the Dirac comb on the spectrum of the sampled signal. We have seen in the lecture about Fourier transform that the spectrum X of a periodic signal x with period T_0 whose Fourier coefficients are $(c_n(x))_{n\in\mathbb{Z}}$ is given by:

$$\forall \omega \in \mathbb{R} \qquad X(\omega) = 2\pi \sum_{n=-\infty}^{+\infty} c_n(x)\delta(\omega - n\omega_0)$$

Proposition 2.1 (Poisson summation formula)
The Fourier transform P_{T_0} of the Dirac comb $p_{T_0}(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT_0)$ is given by

$$\forall \omega \in \mathbb{R} \qquad P_{T_0}(\omega) = \frac{2\pi}{T_0} \sum_{n=-\infty}^{+\infty} \delta(\omega - n\omega_0) = \omega_0 \sum_{n=-\infty}^{+\infty} \delta(\omega - n\omega_0)$$



PROOF : The Dirac comb p_{T_0} is a periodic signal with period T_0. Denote $(c_n(p_{T_0}))_{n\in\mathbb{Z}}$ its Fourier coefficients. We have:

$$\forall n \in \mathbb{Z} \qquad c_n(p_{T_0}) = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} p_{T_0}(t)e^{-in\omega_0 t}dt = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \delta(t)e^{-in\omega_0 t}dt = \frac{1}{T_0}e^{-in\omega_0 0} = \frac{1}{T_0}$$

because δ_0 is the only element of the infinite sum in the interval $\left[-\frac{T_0}{2}, \frac{T_0}{2}\right]$. Since the Dirac comb is periodic, its Fourier transform is written:

$$\forall \omega \in \mathbb{R} \qquad P_{T_0}(\omega) = 2\pi \sum_{n=-\infty}^{+\infty} c_n(p_{T_0})\delta(\omega - n\omega_0) = \frac{2\pi}{T_0}\sum_{n=-\infty}^{+\infty} \delta(\omega - n\omega_0)$$ ∎

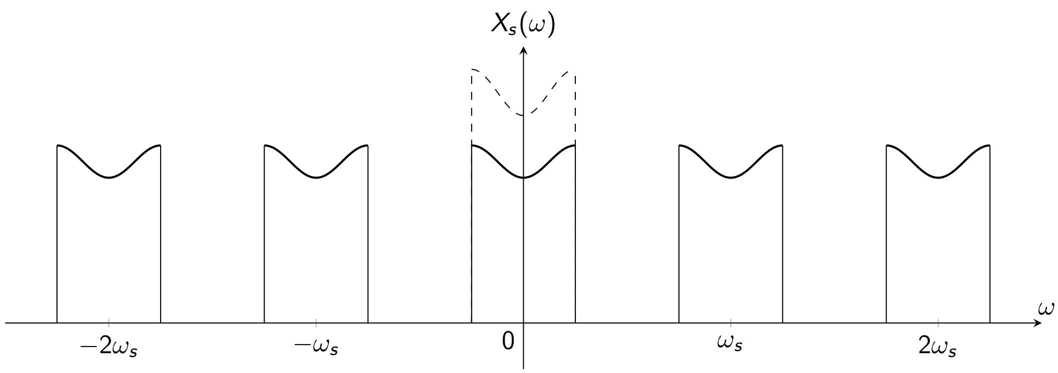
Proposition 2.2
Let an analog signal x and its Fourier transform X. The Fourier transform X_s of the sampled signal $x_s(t) = x(t)p_{T_s}(t)$ is given by:

$$\forall \omega \in \mathbb{R} \qquad X_s(\omega) = \frac{1}{T_s}\sum_{n=-\infty}^{+\infty} X(\omega - n\omega_s)$$

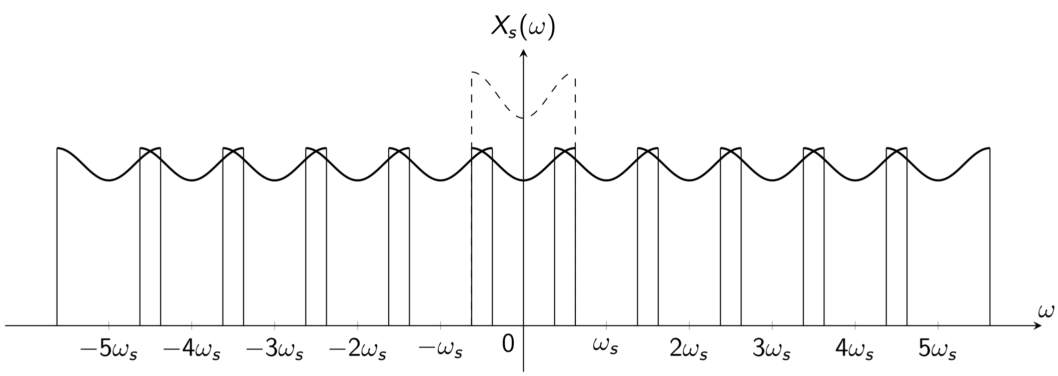
PROOF : Using the Fourier transform of a time product and Poisson summation formula, we get:

$$X_s = \frac{1}{2\pi}(X * P_{T_s}) = \frac{1}{2\pi}\left(X * \frac{2\pi}{T_s}\sum_{n=-\infty}^{+\infty}\delta_{n\omega_s}\right) = \frac{1}{T_s}\sum_{n=-\infty}^{+\infty} X * \delta_{n\omega_s} = \frac{1}{T_s}\sum_{n=-\infty}^{+\infty}\tau_{n\omega_s}(X)$$

which yields the result. ■



Spectral effect of sampling



Sampling overlap

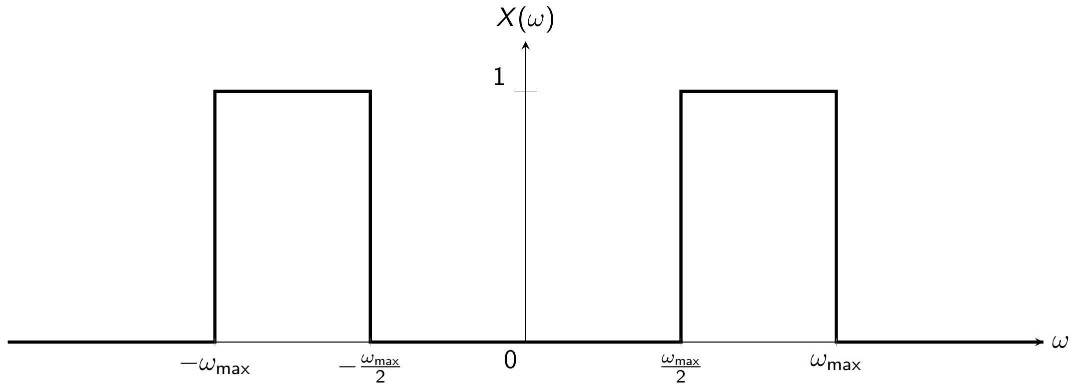
Remark: Graphically, multiplying an analog signal by a Dirac comb boils down to duplicate its spectrum, place a copy in $n\omega_s$ for every $n \in \mathbb{Z}$, and divide the modulus of the spectrum by T_s. It is clear from their definition that the larger the sampling period the smaller the sampling frequency, and vice versa. Thereby, if the sampling period is too large, i.e. the sampling frequency is too small, copies of the original spectrum may intersect, resulting in **spectral overlap** or **aliasing**. The main issue with such an overlap is a loss of information about the spectrum of the original analog signal, which might prevent its perfect reconstruction. The following theorem states a sufficient condition to avoid this overlap.

> **Theorem 2.3 (Shannon-Nyquist sampling theorem)**
> Let an analog frequency-limited signal x whose spectrum X is zero outside the interval $[-\omega_{\max}, \omega_{\max}]$. To prevent any spectral overlap, it is sufficient to sample signal x with a sampling frequency at least twice larger than the maximum frequency in X, i.e. $\omega_s > 2\omega_{\max}$.

PROOF : The previous remark and figures graphically justify this theorem. ■

Remarks:

- This sampling theorem provides a sufficient but not necessary condition to prevent spectral overlap. Indeed, some narrowband spectra can be sampled with a frequency smaller than $2\omega_{\max}$ while avoiding overlap. For instance, the following spectrum can be sampled with frequency $\omega_s = \omega_{\max}$:



- Note that the inequality is strict in this theorem. Sampling with frequency $\omega_s = 2\omega_{\text{max}}$ may not allow perfect reconstruction of the analog signal. For example, if we sample the sine signal $x(t) = \sin(\omega_0 t)$ with frequency $\omega_s = 2\omega_0$, we get, for any $n \in \mathbb{Z}$, $x[n] = \sin(\omega_0 n T_s) = \sin(n\pi) = 0$, which yields the same exact result as the sampling of the zero function.

- We have seen in the lecture about frequency design that the range of sound frequencies audible by the human ear is typically between 16 Hz and 16 kHz. Many digital audio file formats sample audio signals at the sampling frequency of 44.1 kHz. The previous theorem shows that this sampling frequency prevents most of the loss of information in audio signals heard by humans.

- We have seen in the lecture about the time-frequency duality that time-limited signals, which we usually deal with in practice, cannot be frequency-limited as well, thus spectral aliasing is inevitable. To prevent such an overlap, we can preprocess the signal by using a lowpass filter, also known as an **anti-aliasing filter**, to remove the high frequencies causing overlap.